

August 27, 2000

Pappus of Alexandria, (fl. c. 300-c. 350)¹

1 Introduction

Very little is known of Pappus' life. Moreover, very little is known of what his actual contributions were or even exactly when he lived. We do know that he recorded in one of his commentaries on the *Almagest*² that he observed a solar eclipse on October 18, 320. He is regarded, though, as the last great mathematician of the Hellenistic Age. At this time higher geometry was in complete abeyance until Pappus. From his descriptions, we may surmise that either the classical works were lost or forgotten. His self-described task is to 'restore' geometry to a place of significance.

2 Pappus' Work

Toward this end wrote *The Collection* or *The Synagogue*, an extant treatise on geometry which we discuss here and several commentaries, now all lost except for some fragments in Greek or Arabic. One of the commentaries, we note from Proclus, was on *The Elements*. Basically, *The Collection* is a treatise on Geometry, which included everything of interest to him. Whatever explanations or supplements to the works of the great geometers seemed to him necessary, he formulated them as lemmas.

The first published translation into Latin was made by Commandinus in 1589. Others including Eisenmann, John Wallis added to the translations. Friedrich Hultsch gave the definitive Greek text with Latin translation in 1876-8.

Features:

- It is very broad, designed to revive classical geometry.

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²Claudius Ptolemy(100-178 AD) wrote the *Mathematical Collection*, later called *The Almagest*, from the Arabic 'al-magisti' meaning "the greatest."

- It is a guide or handbook to be read *with* the **Elements** and other original works.
- Alternative methods of proof are often given.
- The work shows a thorough grasp of all the subjects treated, independence of judgment, mastery of technique; the style is terse but clear. Pappus is an accomplished and versatile mathematician.
- The range of names of predecessors is immense. In some cases, our only knowledge of some mathematicians is due to his citation. Among many others he mentions Aristaeus the elder, Carpus of Antioch, Conon of Samos, Demetrius of Alexandria, Geminus, Menelaus, and of course the masters.

Summary of Contents:

- **Book I** and first 13 (of 26) propositions of **Book II**. Book II was concerned with very large numbers – powers of myriads (i.e. 10,000).
- **Book III** begins with a summary of finding two mean proportionals ($a : x = x : y = a : y$) between two straight lines. In so doing, he gives the solutions of Eratosthenes, Nicomedes, and Heron. He adds solutions of his own that resembles closely that of Eutocius of Sporus. He also defines *plane* problems, *solid* problems, and *linear* problems, of which the two mean proportionals problem is of the latter type. Pappus
 - Distinguishes (1) *plane* problems, solvable with straight edge and compass
 - Distinguishes (2) *solid* problems, requiring the conics for solution, e.g. solving certain cubics.
 - Distinguishes (3) *linear* problems, problems invoking spirals, quadratrices, and other higher curves
 - Gives a constructive theory of means. That is, given any two of the numbers a, b, c and the type of mean (arithmetic, geometric, or harmonic), he constructs the third.

- Describes the solution of the three famous problems of antiquity, asserts these are not plane problems $\sim 19^{th}$ century.
- Treats the trisection problem, giving another solution involving a hyperbola and a circle.
- Inscribes the five regular solids in the sphere.

In this book Pappus goes to some length to distinguish theorems from problems. Citing the apparent fact that his predecessors combined them as one, he separates those statements calling for a construction as problems, and statements that call upon hypotheses to draw implications as theorems. It is not incumbent on the problem originator to know whether or not the construction can be made. It is the solver's task to determine appropriate conditions for solution.

Pappus goes to some length in his study of the three classic means of antiquity, the arithmetic, the geometric, and the harmonic. Recall the chapter on Pythagoras

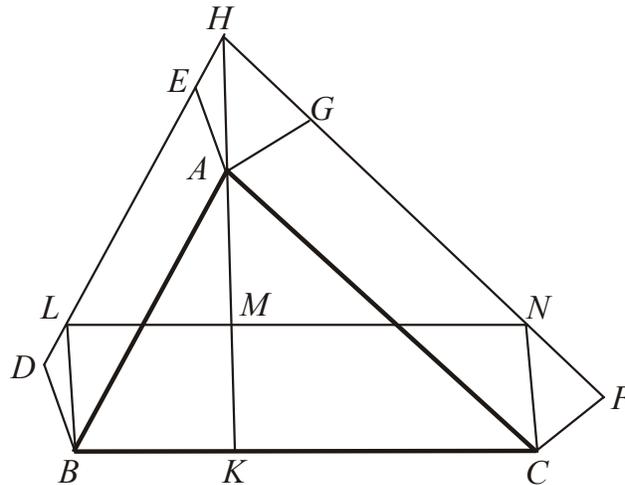
$$\begin{array}{ll}
 a : c = a - b : b - c & \text{Harmonic} \\
 a : a = a - b : b - c & \text{Arithmetic} \\
 a : b - a - b : b - c & \text{Geometric}
 \end{array}$$

where b is the designated mean of a and c . He offers geometric solutions of each. Precisely, for any two of the quantities he constructs the third.

- **Book IV** covers a variety of geometrical propositions. Foremost it contains an extension of theorem of Pythagoras for parallelograms constructed on the legs of any triangle. This result has itself a variety of generalization, and seems to reveal the essence of the Pythagorean theorem itself.

Generalization of Pythagorean's Theorem If ABC is a triangle and on AB, AC any parallelograms are drawn as $ABDE$ and $ACFG$, and if DE and FG are extended to H and HA be joined to K . Then $BCNL$ is a parallelogram and

$$\text{area } ABDE + \text{area } ACFG = \text{area } BCNL$$



Proof. The proof is similar to the original proof of the Pythagorean theorem as found in *The Elements*. First, BL and NC are defined to be parallel to HK . $BLHA$ is a parallelogram and $CAHM$ is a parallelogram. Hence $BCNL$ is a parallelogram.

By “sliding” DE to HL , it is easy to see that

$$\text{area } BDEA = \text{area } BLHA,$$

and by sliding HA to MK it follows that

$$\text{area } BKML = \text{area } BLHA.$$

Thus

$$\text{area } BDEA = \text{area } BKML.$$

Similarly,

$$\text{area } ACFG = \text{area } KCNM.$$

Putting these conclusions together gives the theorem

$$\text{area } ABDE + \text{area } ACFG = \text{area } BCNL.$$

Note. Both parallelograms need not be drawn outside ABC .

Also in Book IV we find material about the Archimedian spiral, including methods of finding area of one turn — differs from Archimedes.

He also constructs the **conchoid** of Nicomedes. In addition, he constructs the quadratrix in two different ways, (1) using a cylindrical helix, and (2) using a right cylinder, the base of which is an Archimedian spiral.

He considers the three problems of antiquity, alluding to them as “solid” problems.³ He offers two solutions of the trisection problem, both involving the use of hyperbolas.

- **Book V** Here we see in the introduction his comments on the sagacity of bees. This statement on the **bees** celebrates the hexagonal shape of their honeycombs.

[The bees], believing themselves, no doubt, to be entrusted with the task of bringing from the gods to the more cultured part of mankind a share of ambrosia in this form, . . . do not think it proper to pour it carelessly into earth or wood or any other unseemly and irregular material, but, collecting the fairest parts of the sweetest flowers growing on the earth, from them they prepare for the reception of the honey the vessels called honeycombs, [with cells] all equal, similar and adjacent, and hexagonal in form.

That they have contrived this in accordance with a certain geometrical forethought we may thus infer. They would necessarily think that the figures must all be adjacent one to another and have their sides common, in order that nothing else might fall into the interstices and so defile their work. Now there are only three rectilinear figures which would satisfy the condition, I mean regular figures which are equilateral and equiangular, inasmuch as irregular figures would be displeasing to the bees. . . . [These being] the triangle, the square and the hexagon,

³An obvious conclusion here is that by this time the ancients generally believed that no classical compass and straight edge constructive solution of these problems was possible.

the bees in their wisdom chose for their work that which has the most angles, perceiving that it would hold more honey than either of the two others.

Bees, then, know just this fact which is useful to them, that the hexagon is greater than the square and the triangle and will hold more honey for the same expenditure of material in constructing each.

In his recounting of various propositions of Archimedes *On the Sphere and Cylinder* he gives geometric proofs of the two familiar trigonometric relations of which the most familiar is

$$\sin(x + y) + \sin(x - y) = 2 \sin x \cos y$$

He also reproduces the work of Zeodorus on isoperimetric figures. He includes the following result.

Proposition. Of all circular segments having the same circumference the semicircle is the greatest.

On solids we find the thirteen semi-regular solids discovered by Archimedes. We also see a number of isoperimetric results such as

Proposition. The sphere is greater than any of the regular solids which has its surface equal to that of the sphere. The proof is similar to that of Zenodorus. He also shows

Proposition. Of regular solids⁴ with surfaces equal, that is greater which has more faces.

- **Book VI** determines the center of an ellipse as a perspective of a circle. It is also astronomical in nature. It has been called the “Little Astronomy”. It covers optics – reflection and refraction.
- **Book VII**, the ‘Treasury of Analysis’ is very important because it surveys a great number of works on geometric analysis of loci, nearly all of which are lost. Features:

– The Book begins with a definition of *analysis* and *synthesis*.

⁴Recall, the tetrahedron, cube, octahedron, dodecahedron, and the icosahedron are the five regular solids. And there are no more, as established during the time of Plato by Theaetetus

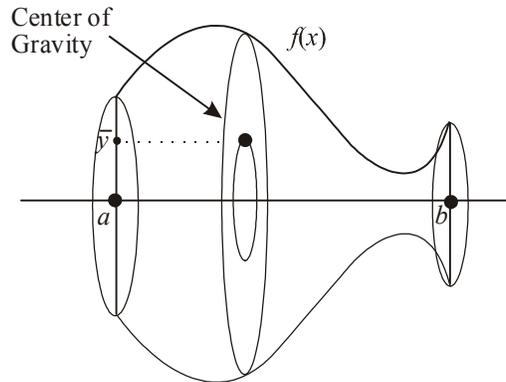
- * *Analysis*, then takes that which is sought as if it were admitted and passes from it through its successive consequences to something which is admitted as the result of synthesis. Unconditional controvertability required.
- * In *Synthesis*, reversing the process, we take as already done that which was last arrived at in the analysis and, by arranging in their natural order as consequences what before were antecedents, and successively connecting them one with the other, we arrive finally at the construction of what was sought.
- A list of the books forming the ‘treasury’ is included, together with a short description of their contents.
- As an independent contribution Pappus formulated the volume of a solid of revolution, the result we now call the The Pappus – Guldin Theorem. P. Guldin (1577-1643)
- Most of the remaining of the treatise is collections of lemmas that will assist the reader’s understanding of the original works.

Pappus also discusses **the three and four lines** theorem of Apollonius.

- Succinctly, given three lines: Find the locus of points for which the product of the distances from two lines is the square of the distance of the third. (The solution is an ellipse.)
- Given four lines: Find the locus of points for which the product of the distances from two lines is the product of the distance of the other two.

Pappus' Theorem.

Volume of revolution = (area bounded by the curve)
 × (distance traveled by the center of gravity)



Volume of revolution $V = \int_a^b \pi f^2(x) dx$

Area bounded by the curve:

$$A = \int_a^b f(x) dx$$

The \bar{y} center of gravity:

$$\bar{y} = \frac{\frac{1}{2} \int_a^b y f(x) dx}{\int_a^b f(x) dx} = \frac{\int_a^b f^2(x) dx}{\int_a^b f(x) dx}$$

Pappus:

$$V = 2\pi\bar{y}A$$

Pappus', on the Pappus-Guldin Theorem

'Figures generated by a complete revolution of a plane figure about an axis are in a ratio compounded (1) of the ratio of the areas of the figures, and (2) of the ratio of the straight lines similarly drawn to (i.e. drawn to meet at the same angles) the axes of rotation from the respective centres of gravity. Figures generated by incomplete revolutions are in the ratio compounded (1) of the ratio of the areas of the figures and (2) of the ratio of the arcs described by the centres of gravity

Pappus

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of the respective figures, the latter ratio being itself compounded (a) of the ratio of the straight lines similarly drawn (from the respective centres of gravity to the axes of rotation) and (b) of the ratio of the angles contained (i.e. described) about the axes of revolution by the extremities of the said straight lines (i.e. the centres of gravity).'

Pappus' theorem surface area.

3 End Game — The End of the Greek School

Following Pappus was no mathematician with abilities of the great masters. The school at Alexandria was diminished with only an occasional bright star yet to shine. The world was turning to Christianity, which at that time in no way resembled what it would become centuries later. To the Christians, the ancient schools were pagan, and paganism must be destroyed.

3.1 Theon of Alexandria

Theon of Alexandria (c. 390) lived toward the end of the period when Alexandria was a center of mathematics. His most valuable contributions are his commentaries on various of the masterpieces, now growing old in libraries. One commentary was on Ptolemy's *Syntaxis* in eleven books. In it we learn of the Greek use of sexagesimal fractions, arithmetic, and root extraction. Theon also wrote commentaries on Euclid's *Optics*.

More significantly, Theon wrote commentaries on Euclid's *Elements*. It appears that his effort was not directed toward the production of an authoritative and accurate addition. (Remember, 700 years have past since it was written.) Rather, Theon seems to have been intent on making what he regarded as improvements. According to Heath⁵, he made alterations where he thought mistakes appeared, he altered some passages too hastily, he made emendations to improve the linguistic form of Euclid, he added explanations to the original, by adding or altering propositions as needed, and he added intermediate steps to Euclid's proofs to assist student understanding. In summary, his intent was to make the monumental work more accessible. We know all of this only because of the discovery of the earlier non-Theonine edition in the Vatican.

⁵Thomas Heath, *A History of Greek Mathematics, II*, Dover, New York, 1921 page 527. This is a reprinting of the original published by the Clarendon Press, Oxford in 1920.

3.2 Hypatia

(c. 370-418) By 397 Christianity became the state religion of the Roman empire and paganism “pagamni” was banned. The Alexandrian school was considered a center of pagan learning and became at risk. Hypatia, daughter of Theon of Alexandria, became a leader of the neoplatonic school⁶ and was so eloquent and persuasive in her beliefs that she was feared a threat to Christianity. In consequence, she was slain in 418 by a fanatical mob led by the Nitrian monks when she refused to repudiate her beliefs. Some accounts argue that this may have resulted because of a dispute between the Roman prefect Orestus and the patriarch bishop Cyrillus (later St. Cyril. Regardless of the cause of her death, the surrounding events and political hostility resulted in the departure of many scholars from Alexandria.

Her mathematical contributions are not well known and indeed are all lost. It has been surmised by statement of Suidas that she wrote commentaries on Diophantus’ *Arithmetica*, Apollonius’ *On Conics*, and possibly on Ptolemy astronomical works. After Hypatia, Alexandrian mathematics came to an end, though there is evidence that little remained at this late date. For further resources on Hypatia, see <http://www-groups.dcs.st-and.ac.uk/history/Mathematicians/Hypatia.html>.

3.3 Eutocius of Ascalon

Eutocius (c. 480 - c. 540), likely a pupil of Ammonius in Alexandria⁷, probably became head of the Alexandrian school after Ammonius. There is no record of any original work by Eutocius. However, his

⁶One unfortunate tenet of neoplatonism, the last school of Greek philosophy, was its declaration of ideological war against the Christians. Created by the great philosopher Plotinus, and carefully edited and promoted by Porphyry (c. 234 - c. 305), neoplatonism featured an extreme spiritualism and a greater sympathy with the less sharply defined hierarchies of the Platonists. Porphyry, who incorporated Aristotle’s logic into neoplatonism, also attacked Christian doctrines on both a philosophical and exegetical basis. Antagonism developed. Interestingly, Porphyry, Iamblichus (c. 250 - c. 330), and much later Proclus (410 - 485) were all prominent neoplatonists and more importantly (for us) were capable mathematicians. Porphyry wrote a commentary on the *Elements* Iamblichus wrote a commentary on Nicomachus’ *Introductio arithmetica*, and Proclus we will study in the next section. Interestingly, until modern times, mathematicians were often also philosophers or clerics. One of the deepest philosopher-mathematicians was René du Perron Descartes (1596 - 1650).

⁷who in turn was a student of Proclus

commentaries contain much historical information which might otherwise have been completely lost. In particular, he wrote commentaries of Archimedes' *On the sphere and cylinder* that inspired some brief interest in his great mathematical works resulting in translations into a more familiar dialect and more suitable for students. A similar result held for his commentaries of Apollonius' *On Conics*.

3.4 Athens

Ironically, the center of Greek mathematics returned to Greece for the first time in nearly 1000 years. The Academia of Plato, which had access to its own ample financial means, maintained itself for a longer time. Proclus Diadochus (411 - 485), though he studied for a brief time in Alexandria, later moved to Athens where he wrote a most important work, namely his commentaries on *Elements*, Book I. Because of the wealth of information he included, it is now one of our main sources of information on the history of geometry. Evidently, Proclus had access to a library of considerable resources including the *History of Geometry* by Eudemus (fl. 335) and other great works. Proclus, a neoplatonist, was venerated even in his own time as being a man of great learning.

After came **Isidore of Alexandria** and **Damascius of Damascus**. They were heads of the school. There was also **Simplicius**, who wrote commentaries on Aristotle. But in 529, on the order of the Emperor Justinian, the school of Athens, the last rampart of the pagan world, was closed.

The last center of Greek culture was **Constantinople**. Here lived **Isidore of Milete** and **Anthemius of Tralles**, both architects and mathematicians. It was probably Isidore who added the so-called 15th Book of the Elements, which contains propositions on regular polyhedra. At least, the propositions were probably his. After these last flutterings, the history of Greek mathematics died.

4 The Decline of Greek Mathematics

Why did mathematics decline so dramatically from the Golden Age? No doubt an entire chapter could be devoted to this topic, even books with carefully crafted answers could be offered up. However, among the main points that could be argued are these:

- There were always only a few that could afford to spend their lives pursuing mathematics. Mathematics, in particular geometry, was clearly now at a level that demanded professional practitioners. It would have to spring in new directions to gather around it a loyal cadre of dedicated amateurs that could sustain the few professionals and feed their numbers.
- The teaching tradition diminished partly due to the political strife around the eastern Mediterranean.
- Roman influence (never inclined to mathematics) was important.
- Arab hegemony – destruction of the library of Alexandria – and of the seat of learning.
- Christian intolerance and the unfortunate classification of mathematics as a “pagan art”.